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A Characterization of Uniquely Ergodic Interval Exchange Maps in Terms of the Jacobi-Perron Algorithm

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Abstract. An interval exchange map T satisfies the infinite distinct orbit condition if the T-orbits of the T-discontinuities are infinite and distinct. We characterize among the interval exchange maps that satisfy this condition those that are uniquely ergodic by the convergence of an associated multidimensional continued fraction in the sense of Jacobi and Perron.

0. introduction

Suppose that for some irrational $\alpha \in (0,1)$, T is the interval exchange map that acts on $[0,1)$ by exchanging the intervals $[0,\alpha)$ and $[\alpha,1)$.

We say that n is a *critical left (or right) iterate* if $T^n(0) < \alpha$ (or $T^n(0) > \alpha$, respectively) and there is no point of the orbit $\{T^1(0), \dots,$ $\dots, T^{n-1}(0)\}$ that is strictly between $T^n(0)$ and the T-discontinuity α. By writing L for every left critical iterate and R for every right critical iterate we associate an infinite L, R word to the T-orbit of 0. By amalgamating successive letters that are the same the word will be of the form $L^{n_1}R^{n_2}L^{n_3}R^{n_4}\cdots$, where $n_1 \geq 0$ and $n_k > 0$, for $k \geq 2$. We have

$$\frac{\alpha}{1-\alpha} = [n_1, n_2, n_3, \cdots] = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cdots}}$$

[**Ba2**] contains a proof of this well known fact that can serve as a motivation for what we do here. Indeed, the main result of this paper generalizes this assertion to interval exchange maps that permute $m \geq 2$

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intervals using multidimensional continued fractions in the sense of Jacobi ([**Ja**]) and Perron ([**Pe**]).

Roughly, the Jacobi-Perron algorithm in dimension $N \geq 1$ is a rule that transforms a sequence of vectors in $\mathbb{N}_0^N$, where $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$, into a given non-negative vector $X \in \mathbb{R}^N$. We then say that the sequence of integer vectors is the N-ary continued fraction expansion of X (see section 3). Contrary to the case of ordinary continued fractions ($N = 1$), a given $X \in \mathbb{R}^N$, for $N > 1$, allows many different expansions (which is rather an advantage), and convergence of the partial quotients for a given sequence of integer vectors is by no means guaranteed (which is the main difficulty).

Suppose that for some positive vector $(\alpha_1, \ldots, \alpha_m)$, that satisfies $\alpha_1 + \cdots + \alpha_m = 1$, T is the interval exchange map that permutes the intervals $[0, \alpha_1)$, $[\alpha_1, \alpha_1 + \alpha_2)$, e.t.c. according to a given permutation π (section 1). We say that n is a left or right critical iterate of T if the definition given in the beginning of the section for $m = 2$ holds for some T-discontinuity (section 2.1). We associate in this way as before a L, R-sequence to the T-orbit of 0.

Suppose now that T satisfies Keane's infinite orbit condition (i.d.o.c), which means that the T-orbits of the T-discontinuities are infinite and distinct. For some conveniently chosen critical iterate n, we represent T by a stack $S(g, X)$, where g represents the combinatorial type of the stack and X is an integer vector (section 2.3). We then transform the L, R sequence that is associated to the T-orbit of 0 into a sequence of integer vectors of dimension N, the N-ary continued fraction associated to X (not *of* X, as convergence is a problem), where $N = 2m - 3$ (section 4).

Our main result is theorem 4 of section 4 which states that an i.d.o.c. interval exchange map is uniquely ergodic if and only if the N-ary continued fraction expansion associated to X converges to X.

The result is constructive. Indeed, for a given permutation π on m symbols one can construct the finite number of possible combinatorial types (described in section 2.2). To a given combinatorial type and a letter L or R there is associated another combinatorial type (section 2.4) and a finite sequence of integer vectors (section 4). This information is easy to determine and only depends on π. So starting with g as described above, we apply successively the letters of the L, R-sequence that is associated to the T-orbit of 0 to get a sequence of combinatorial types and hence an infinite sequence of integer vectors.

For the reader who wishes a stronger analogy between the cases of an interval exchange map of $m = 2$ and $m > 2$ intervals, we propose two possibilities: Either we write in case $m = 2$ the L, R sequence as $L \cdots LR \cdots R \cdots$ and replace each L by the finite sequence $1, 0$ and each R by $0, 1$ to get an infinite sequence whose terms are in $\mathbb{N}_0$. The obvious rules

$$[\cdots, n, 0, 0, m, \cdots] = [\cdots, n, m, \cdots]$$

and

$$[\cdots, n, 0, m, \cdots] = [\cdots, n + m, \cdots]$$

show that this procedure yields the same result as the one mentioned in the beginning of this section. Or we amalgamate successive letters L (or R) in the L, R-sequence in case $m > 2$ that correspond to a left (or right, respectively) critical iterate of the same type (see section 2.1 for the definition) to get a sequence of the form $L^{n_1} \cdots L^{n_k} R^{m_1} \cdots R^{m_s} \cdots$. To determine the finite sequence of vectors that correspond to a letter with its power we determine the finite sequence in case the power is 1 as before and then multiply each vector by the power.

This paper relies heavily on a paper by Rocha ([**Ro**]). In particular, an adaptation of the characterization of uniquely ergodic i.d.o.c. interval exchange maps given there is crucial to prove convergence of the continued fractions mentioned above. Most of section 2 is a collection of results from [**Ro**] that are used in this paper.

This paper is a sequel to [**Ba2**] where we expressed the topological entropy and the invariant measured foliations of pseudo-Anosov maps in multidimensional continued fraction form using interval exchange maps. As the foliations in question and hence the corresponding interval exchange maps are uniquely ergodic, theorem 4 of section 4 also gives a continued fraction expansion of the invariant foliations. However, the methods used in [**Ba2**] are better adapted to the geometric problem in question.

1. Interval exchange maps

Throughout this paper we fix a number $m \geq 2$ and a permutation π of $\{1, \ldots, m\}$. We will always use the norm of $x = (x_1, \ldots, x_N) \in \mathbb{R}^N$ that is defined by $||x|| = \sum_{i=1}^{N} |x_i|$.

Each $\alpha = (\alpha_1, \ldots, \alpha_m)$ in

$$S_m = \{\beta = (\beta_1, \ldots, \beta_m) \in \mathbb{R}^m \mid \beta_i > 0, \text{ for } i \in \{1, \ldots, m\} \text{ and } ||\beta|| = 1\}$$

defines an interval exchange map $T = T(\pi, \alpha)$ that acts on $[0, 1)$ as follows: decompose $[0, 1)$ into half-open intervals $I_1 = I_1(T), \ldots, I_m = I_m(T)$ of respective lengths $\alpha_1, \ldots, \alpha_m$, such that the initial point of I_1 is 0, the initial point of I_2 is $D_1 = D_1(T) = \alpha_1$, the initial point of I_3 is $D_2 = D_2(T) = \alpha_1 + \alpha_2$, e.t.c. The restriction of T to each I_i is an Euclidean isometry. Moreover, starting from 0 we encounter in succession the half-open intervals $T(I_{\pi^{-1}(1)})$, $T(I_{\pi^{-1}(2)})$, e.t.c. As we fix π we can, and often will, identify $T(\pi, \alpha)$ with α.

Keane ([**Ke**]) showed that almost any interval exchange map satisfies the *infinite distinct orbit condition (i.d.o.c)* which means that for each $i \in \{1, \ldots, m-1\}$ the T orbit of the discontinuity D_i is infinite and that for different i and j the corresponding orbits are disjoint. Moreover, he showed that an interval exchange map T that satisfies this condition is *minimal*, i.e. the positive orbit of any point is dense in $[0, 1)$.

The main goal is to characterize among the interval exchange maps that satisfy i.d.o.c. those that are *uniquely ergodic*, i.e. those whose only invariant Borel measure is Lebesque measure. It was shown by Masur ([**Ma**]), Veech ([**Ve**]) and later by Lopes-Rocha ([**LR**]) that almost any interval exchange map is uniquely ergodic.

It is clear that we only need to consider interval exchange maps whose underlying permutation π is *discontinuous* (i.e. $\pi(i)+1 \neq \pi(i+1)$, for $i \in \{1, \ldots, m\}$), as otherwise we are in the situation of an interval exchange map that permutes fewer than m intervals. Moreover, an obvious necessary condition for minimality and hence unique ergodicity is the *irreducibility* of π (i.e. $\pi(\{1,\dots,k\}) = \{1,\dots,k\}$ only if $k = m$).

We suppose throughout that our (fixed) permutation π is discontinuous and irreducible.

2. Farey cells

For the results of this section that are mentioned without proof we refer the reader to the paper by Rocha ([**Ro**]).

For a given $n \in \mathbb{N}$, we say that the interval exchange maps T and S are *equivalent up to the order* n, if for $k \in \{1,\dots,n\}$ we have $T^k(0) \in I_i(T)$, where $i = i(k)$, if and only if $S^k(0) \in I_i(S)$, for the same i. The equivalence class of T (with respect to n) is called the *Farey cell of order* n *centered at* T, $\mathcal{F}_n = \mathcal{F}_n(T)$. $\mathcal{F}_n$ is a non-empty set whose closure is a convex polyhedron.

We say that a Farey cell $\mathcal{F}_n$ is *small*, if for some (and hence all) $T \in interior\mathcal{F}_n$ each interval $I_i(T)$ and $T(I_i(T))$, for $i \in \{1,\dots,m\}$, contains at least one point of the orbit $\{T^i(0)\}_{i=0}^n$. It is clear that the Farey cell of order n centered at an i.d.o.c. T will be small for n sufficiently large.

We say that n is a *left (or right) critical iterate of type* i_0 of T, if the orbit (as a set) $\{T^i(0)\}_{i=0}^n$ intersects the interval $[T^n(0), D_{i_0}(T))$ (or $[D_{i_0}(T), T^n(0)]$, respectively) only in the point $T^n(0)$. Note in particular that if $T^n(0) = D_{i_0}(T)$ then n is a right critical iterate of type i_0.

If n is a critical iterate of some $T \in interior\mathcal{F}_n$, then it is critical for all other elements of $interior\mathcal{F}_n$, so by abusing notation slightly, we will say that n is a critical iterate of $\mathcal{F}_n$. Moreover, the first critical iterate that follows the critical iterate n is independent of $T \in interior\mathcal{F}_n$.

An interval exchange map has arbitrarily large critical iterates. Moreover, if $s < n$ are two consecutive critical iterates of T and if $T \in interior\mathcal{F}_n$, then $\mathcal{F}_s(T) \supseteq \mathcal{F}_n(T)$, whereas $\mathcal{F}_s(T) = \mathcal{F}_k(T)$ for $k \in \{s,\dots,n-1\}$. We remark in passing that it is possible that $T^s(0) = T^n(0)$, and hence $\mathcal{F}_s(T) = \mathcal{F}_n(T)$.

We will show in section 2.3 that each small Farey cells $\mathcal{F}_s$ with s

critical can be identified with an abstract model to be defined next.

2.2. Abstract Farey cells

Associated to π is a bijection

$$f = f(\pi) : \{0, \dots, m-1\} \to \{1, \dots, m\}.$$

The relevant data for an abstract Farey cell consists of a surjective map

$$g : \{0, \dots, m-1\} \to \{1, \dots, m-1\}$$

and a convex subset C_g of $\mathbb{R}^{2m}$ of dimension $m-1$:
There is exactly one i_0, the *type of* g, in the image of g such that the cardinality of the set $g^{-1}(\{i_0\})$ is 2, namely it contains $g^{m-1}(0)$ and another element that we call $g^{-1}(i_0)$. By defining $g^{-1}(i) = g^{-1}(\{i\})$, for $i \neq i_0$ we get a right inverse of g.

The column vector $X = (L_0, \dots, L_{m-1}, R_1, \dots, R_m)^t$ is in C_g, if and only if

a) $L_0 = R_m = 0$;
b) $L_i > 0$ and $R_i \geq 0$, for $i \in \{1, \dots, m-1\}$;
c) $||X|| = 1$;
d) For $i \in \{1, \dots, m-1\} \setminus \{i_0\}$ we have $L_i + R_i = L_{g^{-1}(i)} + R_{f \circ g^{-1}(i)}$.
Finally,

$$L_{i_0} + R_{i_0} = L_{g^{-1}(i_0)} + R_{f \circ g^{-1}(i_0)} + L_{g^{m-1}(0)} + R_{f \circ g^{m-1}(0)}.$$

We refer to C_g as the *abstract Farey cell of type* g.

2.3. Stacks

Suppose that $\mathcal{F}_s$ is a small Farey cell with s a critical iterate. We show how to identify the elements of $\mathcal{F}_s$ with the elements of an abstract Farey cell C_g.

Suppose that n is the first critical iterate for the elements of $interior \mathcal{F}_s$ that follows s and that $\tilde{T}$ is in the interior of $\mathcal{F}_s$. We choose T in the interior of $\mathcal{F}_n(\tilde{T}) \subset \mathcal{F}_s$, hence the set $O_n = \{T^k(0)\}_{k=0}^n$ contains no T-discontinuity. The points of O_n decompose $[0, 1)$ into a collection of intervals that we may take to be closed and the interval $[M, 1)$, where

$M = sup\{T^k(0) \mid k = 1, \ldots, n\}$. Except for the interval $[M, 1)$, the intervals are bounded below and above by a point of O_n and contain no other point of O_n in its interior. Note that $m-1$ of the intervals will contain (exactly) one point D_i, for $i \in \{1, \ldots, m-1\}$ in its interior.

We now arrange the intervals that make up $[0, M]$ into m stacks by placing an interval I' on top of an interval I, if I is mapped to I' under T in such a way that $m-1$ of the stacks contain a point of discontinuity of T in the interior of its top interval. Moreover, the top interval of the last stack has $T^n(0)$ as one of its end points but no T-discontinuity in its interior.

We temporarily label the stacks as follows: The stack whose top interval contains $D_i = D_i(T)$ is referred to as Q_i, for $i \in \{1, \ldots, m-1\}$. The top interval of one of those stacks, say Q_{i_0}, has $T^n(0)$ as one of its endpoints. We then denote by Q'_{i_0} the stack whose top interval does not contain a discontinuity of T, but has $T^n(0)$ as one of its endpoints.

For $i \in \{1, \ldots, m-1\} \setminus \{i_0\}$, the discontinuity D_i decomposes the top interval of Q_i into a left interval $L_i^\sharp = [T^{l(i)}, D_i)$ and a right interval $R_i^\sharp = [D_i, T^{r(i)}]$, for some $0 < l(i), r(i) \leq s$. See figure 1 a). The notation used for various points and subintervals of the bottom interval of the stack will be explained presently.

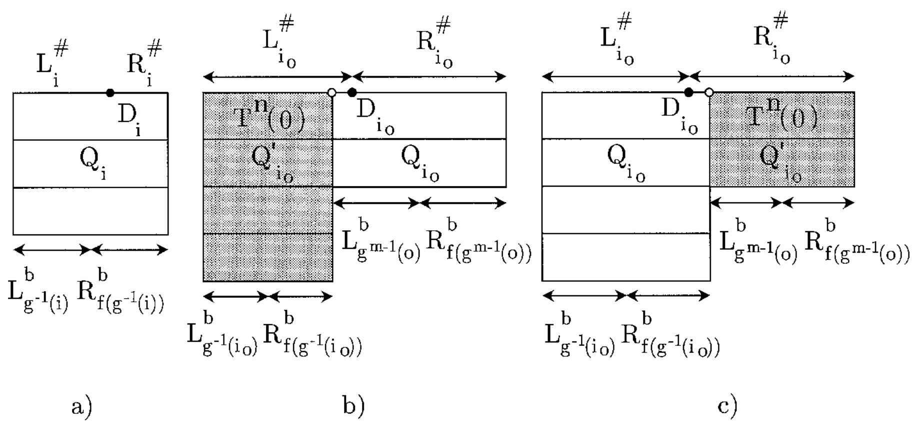

Figure 1: Stack.

We next combine the two stacks Q_{i_0} and Q'_{i_0} so that its top intervals are joined at their common endpoint $T^n(0)$ and form a connected horizontal interval. The point D_{i_0} decomposes this interval into a left interval $L^\sharp_{i_0}$ and a right interval $R^\sharp_{i_0}$. See figure 1 b) in case n is a left critical iterate and figure 1 c) in case n is a right critical iterate. The stack Q'_{i_0} is shaded in the figure.

Suppose that $i \in \{1, \ldots, m-1\}$. In case $T(L^\sharp_i)$ is $[M, 1)$, which is not part of any stack, we have that $T^2(L^\sharp_i)$ is contained in the bottom of a stack. In this case we denote the interval $T^2(L^\sharp_i)$ by L^b_i. Otherwise $T(L^\sharp_i)$ itself is contained in the bottom of a stack in which case we set $L^b_i = T(L^\sharp_i)$.

We similarly denote the interval $T(R^\sharp_i)$ that is contained in the bottom of a stack by R^b_i, for $i \in \{1, \ldots, m-1\}$.

For convenience of notation we introduce the empty intervals L^b_0 and R^b_m. One can see that there is a map g as in the definition of abstract Farey cell, so that the lower side of the stack Q_i consists of the intervals $L^b_{g^{-1}(i)}$ and $R^b_{f\circ g^{-1}(i)}$, for $i \in \{1, \ldots, m-1\} \setminus \{i_0\}$. See figure 1 a). Moreover, the lower side of the stack Q_{i_0} or Q'_{i_0} that was placed to the left of the other is made up of the intervals $L^b_{g^{-1}(i_0)}$ and $R^b_{f\circ g^{-1}(i_0)}$, whereas the lower side of the stack that was placed to the right consists of the intervals $L^b_{g^{m-1}(0)}$ and $R^b_{f\circ g^{m-1}(0)}$. See figure 1 b) and c).

Example 1. As an example we take $\pi = (2, 4, 1, 3)$ and suppose that $T = T(\pi, \alpha)$ is an interval exchange map whose points $\{T^0(0), \ldots, T^{15}(0)\}$ are as indicated in figure 2 a). We take $s = 7$. The first critical iterate that follows s is $n = 10$. The stack that describes T up to the order 7 is shown in figure 2 b). We have $l(1) = 2$, $r(1) = 3$, $l(2) = 5$, $r(2) = 7$, $l(3) = 7$ and $r(3) = 4$. The function g is given by $g(0) = g(3) = 1$, $g(1) = 2$ and $g(2) = 3$. Moreover, $f(0) = 2$, $f(1) = 3$, $f(2) = 1$ and $f(3) = 4$. So $i_0 = 1$ and the bottom interval of Q_1 consists of the empty interval $L^b_{g^{-1}(1)} = L^b_0$ and $R^b_{f(0)} = R^b_2$, whereas the bottom interval of Q'_{i_0} consists of $L^b_{g^3(0)} = L^b_3$ and the empty interval $R^b_{f(3)} = R^b_4$.

We set $L_0 = 0 = R_m$ and denote the length of the interval $L^\sharp_i$ (and $R^\sharp_i$) by L_i (and R_i, respectively), for $i \in \{1, \ldots, m-1\}$. We set $X = (L_0, \ldots, L_{m-1}, R_1, \ldots, R_m)$ and refer to the m stacks as *the collection of stacks* $S(g, X)$ or, if there is no risk of confusion, simply as *the stack*

$S(g, X)$.

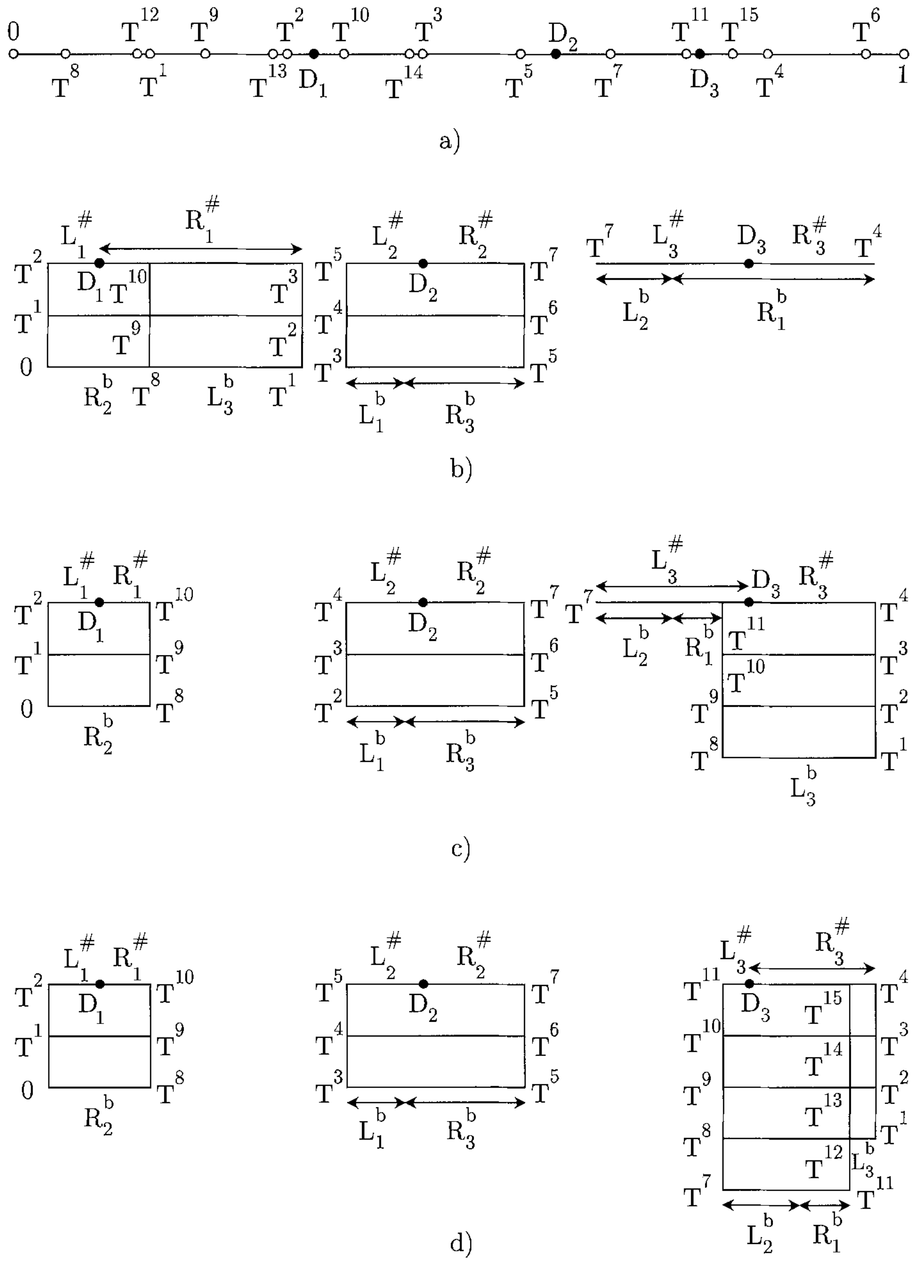

Figure 2: Example 1.

One can show that g is independent of $T \in interior\mathcal{F}_n(\tilde{T})$, so the relation that associates to $T = T(\pi, \alpha) \in interior\mathcal{F}_n$ the corresponding stack $S(g, X)$ induces a map $\mathcal{L}$ defined on $interior\mathcal{F}_n$ that sends α to X. One can see that $\mathcal{L}$ is linear and we denote by the same symbol the extension of $\mathcal{L}$ to all of $\mathcal{F}_n(\tilde{T})$.

Note that for $i \in \{1, \dots, m-1\}$, $L_i = D_i - T^{l(i)}$, where $l(i)$ is the left critical iterate of type i that remains critical up to the order s, i.e. the largest left critical iterate of T of type i that is smaller or equal to s. Similarly, if $i \in \{1, \dots, m-1\}$ then $R_i = T^{r(i)} - D_i$, where $r(i)$ is the right critical iterate of type i that remains critical up to the order s. So X only depends on the orbit of 0 up to T^s, in fact $l(i)$ and $r(i)$, for $i \in \{1, \dots, m-1\}$, are independent of $interior\mathcal{F}_s$. As g is in fact independent of $\tilde{T} \in interior\mathcal{F}_s$, the linear map $\mathcal{L}$ defined on $\mathcal{F}_n(\tilde{T})$ is independent of $\tilde{T} \in interior\mathcal{F}_s$, hence is defined on all of $\mathcal{F}_s$.

If $T = T(\pi, \alpha) \in interior\mathcal{F}_s$ and $X = \mathcal{L}(\alpha)$, then we say that the stack $S(g, X)$ *represents* T *up to the order* s. In this case it is immediate that $\mathcal{L}(\alpha)/||\mathcal{L}(\alpha)||$ is an element of the abstract Farey cell C_g. In fact the map $\mathcal{L}/||\mathcal{L}||$ is a bijection between $\mathcal{F}_s$ and C_g. One can find a linear map M defined on the image of $\mathcal{L}$ that is the inverse of $\mathcal{L}$, hence $M/||M||$, defined on C_g, is the inverse of $\mathcal{L}/||\mathcal{L}||$. The map M, when represented in matrix form with respect to the usual basis of Euclidean space, is referred to as *distribution matrix associated to* $\mathcal{F}_s$.

2.4. The transition matrices

Suppose that $\mathcal{F}_s(\tilde{T})$ is a small Farey cell with s critical. Let n denote the first critical iterate of the elements of $interior\mathcal{F}_s(\tilde{T})$ after s and p the first critical iterate of the elements of $interior\mathcal{F}_n(\tilde{T})$ after n.

We represent a given $T \in interior\mathcal{F}_p(\tilde{T})$ by the stack $S(g, X)$ up to the order s and by the stack $S(\tilde{g}, \tilde{X})$ up to the order n. Suppose that i_0 is the type of g. So, in the notation used in the construction of a collection of stacks, $S(g, X)$ is made up of the stacks $Q_1, \dots, Q_{m-1}, Q'_{i_0}$.

We perform the following operation on the stack $S(g, X)$: The top side of Q'_{i_0} is mapped (by T or T^2) to some interval, which we temporarily call I, in the bottom of some other stack, say Q_j. We separate the stacks Q_{i_0} and Q'_{i_0} and place Q'_{i_0} below Q_j in such a way that the top interval of Q'_{i_0} is below I. We then mark the points $\{T^i(0)\}_{i=n+1}^{p}$ in the intervals that constitute Q_j above the point T^n which is contained in the top interval of Q'_{i_0}. We constructed a new stack which can be seen to be $S(\tilde{g}, \tilde{X})$ (so in particular, j is the type of $\tilde{g}$). We say that we performed a *left cut* if n is a left critical iterate (as in figure 1 b) and a *right cut* if n is a right critical iterate (as in figure 1 c).

Example 1. (continuation). By applying the right cut to the stack shown in figure 2 b) we get the stack of figure 2 c) that represents T up to the order 10. Applying the left cut to the stack of figure 2 c) yields the stack of figure 2 d) that represents T up to the order 11.

We set

$$X = (L_0, \dots , L_{m-1}, R_1, \dots , R_m)$$

and

$$\tilde{X} = (\tilde{L}_0, \dots , \tilde{L}_{m-1}, \tilde{R}_1, \dots , \tilde{R}_m).$$

If we get $S(\tilde{g}, \tilde{X})$ from $S(g, X)$ by a left cut, then $\tilde{R}_i = R_i$, for $i \in \{1, \dots , m\}$ and $\tilde{L}_i = L_i$, for $i \in \{0, \dots , m-1\} \setminus \{i_0\}$. Finally, we have $\tilde{L}_{i_0} = L_{i_0} - (L_{g^{-1}(i_0)} + R_{f\circ g^{-1}(i_0)})$. This means that the distribution matrices M and $\tilde{M}$ associated to $\mathcal{F}_s$ and $\mathcal{F}_n$, respectively, can be written as $\tilde{M} = MK$, where K differs from the identity matrix only by containing 1 in two off diagonal positions, namely in the $(i_0 + 1, g^{-1}(i_0) + 1)$ and the $(i_0 + 1, m + f \circ g^{-1}(i_0))$ position.

In case of a right cut we have $\tilde{L}_i = L_i$, for $i \in \{0, \dots , m-1\}$ and $\tilde{R}_i = R_i$, for $i \in \{1, \dots , m\} \setminus \{i_0\}$. Finally, we have $\tilde{R}_{i_0} = R_{i_0} - (L_{g^{m-1}(0)} + R_{f\circ g^{m-1}(0)})$. So the matrix K that is defined as before differs from the identity matrix only by containing 1 in the $(m+i_0, g^{m-1}(0)+1)$ and $(m+i_0, m+f\circ g^{m-1}(0))$ position. We refer to K as *transition matrix.*

Example 1. (continuation). The transition matrix that is associated to the left cut applied to the stack of figure 2 b) differs from the identity matrix only by containing extra entries 1 in positions $(5, 4)$ and $(5, 8)$. The latter accounts for the empty interval R_4 not shown in figure 2 b).

Finally, the transition matrix that describes the right cut applied to the stack of figure 2 c) has an additional entry one in the $(4,3)$ and $(4,5)$ position.

From now on we delete the dummy variables L_0 and R_m in the vector X and also the rows and columns of the distribution and transition matrices that correspond to those two variables.

Suppose next that $\mathcal{F}_s(\tilde{T})$, $\mathcal{F}_n(\tilde{T})$, p, $T \in interior\mathcal{F}_p(\tilde{T})$, $S(g,X)$ and $S(\tilde{g},\tilde{X})$ are as in the beginning of the section, except that we only demand that $\mathcal{F}_n(\tilde{T})$ is a Farey cell with $n > s$ and n critical but *not* that n is the first critical iterate that follows s.

It follows from the above that the distribution matrices M and $\tilde{M}$ associated to $\mathcal{F}_s$ and $\mathcal{F}_n$, respectively, can be written as $\tilde{M} = MK$, where K is a product of transition matrices.

We give a description of K that is similar to the one for distribution matrices given in [**Ro**].

Let $l(i)$ and $\tilde{l}(i)$ be the critical left iterate of type i that remains critical up to the order s and n, respectively. Denote by $\tilde{a}(i)$ the smaller of p and the first critical left iterate that follows $\tilde{l}(i)$. Similarly, let $r(i)$ and $\tilde{r}(i)$ be the critical right iterate of type i that remains critical up to the order s and n, respectively, and denote by $\tilde{b}(i)$ the smaller of p and the first critical right iterate that follows $\tilde{r}(i)$.

We claim:

Lemma 1.
a) *The jth column $\lambda_j = (\lambda_{1,j}, \dots, \lambda_{2m-2,j})^t$, for $j \in \{1, \dots, m-1\}$, of K is given by*

$$\lambda_{i,j} = \#\{k \mid T^k(0) \in [T^{l(i)}(0), D_i) \text{ and } \tilde{l}(j) \le k < \tilde{a}(j)\},$$

if $1 \le i \le m-1$, and

$$\lambda_{i,j} = \#\{k \mid T^k(0) \in [D_{i-m+1}, T^{r(i-m+1)}(0)) \text{ and } \tilde{l}(j) \le k < \tilde{a}(j)\},$$

if $m \le i \le 2m-2$.

b) *The last $m-1$ columns $\rho_j = (\rho_{1,j}, \dots, \rho_{2m-2,j})^t$, for $j \in \{1, \dots, m-1\}$, of K are given by*

$$\rho_{i,j} = \#\{k \mid T^k(0) \in (T^{l(i)}(0), D_i] \text{ and } \tilde{r}(j) \le k < \tilde{b}(j)\},$$

if $1 \le i \le m-1$, *and*

$$\rho_{i,j} = \#\{k \mid T^k(0) \in (D_{i-m+1}, T^{r(i-m+1)}(0)] \text{ and } \tilde{r}(j) \le k < \tilde{b}(j)\},$$

if $m \le i \le 2m-2$.

Proof. We choose the notation for various points and intervals of the stack $S(g, X)$ as in the definition of stack and add a "tilde" to the notation of the corresponding points and intervals of $S(\tilde{g}, \tilde{X})$.

Suppose that $\tilde{K}$ is the matrix as defined in the lemma. To motivate the definition of $\tilde{K}$, note first that $T^{\tilde{a}(i)}$ is the first iterate after $T^{\tilde{l}(i)}$ that is in the top interval of a stack. Each iterate $T^k(0) \in [T^{l(i)}(0), D_i)$ used in the definition of $\lambda_{i,j}$, for $1 \le i, j \le m-1$, determines a unique interval in $L_i^\sharp$ of the same length as $\tilde{L}_j^\sharp$. A similar statement holds for the iterates of $T(0)$ used in the definition of the other entries of $\tilde{K}$. As the union of the top intervals $L_i^\sharp$ and $R_i^\sharp$ are decomposed in this way into intervals of lengths $\tilde{L}_j$ and $\tilde{R}_j$, for $j \in \{1, \dots, m-1\}$, we see that $X = \tilde{K}\tilde{X}$, which implies that $\tilde{M} = M\tilde{K}$.

This does not yet show that $\tilde{K} = K$. To that end we suppose next that p is a left critical iterate of type i_0, perform the corresponding left cut and calculate the matrix K' as defined in the lemma with respect to the new stack. Although there is a new critical iterate of type i_0 the iterates used in the definition of the i_0-row of K' are in the same stack above the interval $\tilde{L}_{i_0}^b$ as the iterates used in the definition of the i_0-row of K, so the rows are identical.

The T-iterates used in the definition of the $g^{-1}(i_0)$-row of K' are those used in the definition of the same row of K plus the iterates used in the definition of the i_0-row. So the $g^{-1}(i_0)$-row of K' is the sum of the $g^{-1}(i_0)$-row and the i_0-row of K.

Similarly, the iterates of $T(0)$ used in the definition of the $(m-1+f \circ g^{-1}(i_0))$-row of K' are those used in the definition of the same row of K plus iterates that visit the same intervals as the iterates used in the definition of the i_0-row.

The other rows of K and $\tilde{K}$ agree.

One readily sees that $K' = \tilde{K}M$, where M is the transition matrix that describes the left cut.

One proceeds similarly in case p is a right critical iterate.

This constitutes the induction step in an obvious inductive argument that proves that the matrix defined in the lemma is a product of transition matrices, proving the claim of the lemma. □

2.5. Unique ergodicity

Suppose that $T = T(\pi, \alpha)$ is an i.d.o.c. interval exchange map whose sequence of critical iterates that follows the first critical iterate $n(1)$ for which the corresponding Farey cell is small is $\{n(k)\}_{k=1}^{\infty}$.

0 is already the image of a T-discontinuity, so the (positive) T-orbit of 0 does not contain a discontinuity of T. We conclude that T is in the interior of each Farey cell centered at T. It follows that for $k \in \mathbb{N}$, $n(k)$ is critical for $\mathcal{F}_{n(k)}$ and $n(k+1)$ is the first critical iterate of the elements of $interior \mathcal{F}_{n(k)}$ after $n(k)$. For each $k \in \mathbb{N}$ we denote the distribution matrix that corresponds to $\mathcal{F}_{n(k)}$ by M_k.

The study of Farey cells is motivated by the fact that the collection of T-invariant Borel probabilities is in bijective correspondence with $\cap_{k=1}^{\infty} interior \mathcal{F}_{n(k)}(T)$.

One can see that $\cap_{k=1}^{\infty} interior \mathcal{F}_{n(k)}(T)$ consists of exactly one element if and only if each column of the sequence of distribution matrices converges projectively to α, i.e. the jth column $h_k^{(j)}$ of M_k satisfies

$$\lim_{k \to \infty} \frac{h_k^{(j)}}{||h_k^{(j)}||} \stackrel{p}{=} \alpha,$$

for each $j \in \{1, \ldots, 2m-2\}$.

As we are more interested in the transition matrices, we will need to adapt these results.

For $k \in \mathbb{N}$ we denote by $S(g_k, X_k)$ the stack that represents T up to the order $n(k)$. For each $k \in \mathbb{N}$ we denote the various points and intervals of $S(g_k, X_k)$ by adding the superscript (k) to the notation given in the definition of stack. For each $k \in \mathbb{N}$ we have a transition matrix K_k that satisfies $M_{k+1} = M_k K_k$, hence $M_{k+1} = M_1 K_1 K_2 \cdots K_k$. We will need:

Proposition 2. *The i.d.o.c. interval exchange map* $T = T(\pi, \alpha)$ *is uniquely ergodic, if and only if each column of* $K_1 \cdots K_k$ *converges pro-*

jectively to X_1, as k tends to infinity.

Proof. Suppose first that each column of the product of the transition matrices converges projectively to X_1. It follows that each column of the sequence of distribution matrices converges projectively to $\alpha = M_1 X_1$, which implies the unique ergodicity of T.

Conversely, we suppose that T is uniquely ergodic and imitate the argument used in [**Ro**] to show that unique ergodicity implies the convergence of the rows of the distribution matrices.

One can show that for each pair of integer sequences $\{a_k\}_{k=1}^{\infty}$ and $\{A_k\}_{k=1}^{\infty}$ such that $\lim_{k\to\infty}(A_k - a_k) = \infty$, the sequence of Borel probabilities

$$\mu_k = \frac{1}{A_k - a_k} \sum_{t=a_k}^{A_k - 1} \delta_{T^t(0)},$$

where δ_z denotes the Borel Dirac measure that is concentrated at z, converges weakly to the Lebesque measure on $[0,1)$.

Fix $j \in \{1, \dots, m-1\}$ and set $a_k(j) = l^{(k)}(j)$, i.e. the critical left iterate of type j that remains critical up to the order $n(k)$. Denote by $A_k(j)$ the smaller of $n(k+1)$ and the next critical left iterate. For $i \in \{1, \dots, m-1\}$, we calculate the μ_k measure of $Y_i = [T^{l^{(1)}(i)}(0), D_i)$, to be

$$\int \chi_{Y_i} d\mu_k = \frac{1}{A_k(j) - a_k(j)} \sum_{t=a_k(j)}^{A_k(j)-1} \chi_{Y_i}(T^t(0)).$$

The right hand side of the above equation is the ith entry of the jth column of $K_1 \cdots K_k$, whereas the left hand side converges to the length of the interval Y_i which is $L_i^{(1)}$. By determining the μ_k measure of $Z_i = [D_i, T^{r^{(1)}(i)}(0))$, for $i \in \{1, \dots, m-1\}$, we find an equation whose left hand side converges to $R_i^{(1)}$ and whose right hand side is the $(m-1+i)$th entry of the jth column of $K_1 \cdots K_k$.

We showed that the jth column of $K_1 \cdots K_k$ converges projectively to X_1, for $j \in \{1, \dots, m-1\}$.

Using the right critical iterates, a similar argument shows that each of the last $m-1$ columns of $K_1 \cdots K_k$ converges projectively to X_1, finishing the proof of the proposition. □

3. The Jacobi-Perron algorithm

For more information on the Jacobi-Perron algorithm we refer the reader to [**Hu**] and the monograph by [**Be**].

To motivate the definition of the Jacobi-Perron algorithm we recall the definition of the usual continued fractions.

For $x, y \in \mathbb{R}^N$, we write $x \overset{p}{=} y$, if $x = cy$, for some strictly positive $c \in \mathbb{R}$.

If $(n_i)_{i=1}^{\infty}$ is a sequence of positive integers, then we say that the continued fraction expansion of $x > 1$ is $x = [n_1, \dots, n_k, \dots]$ if

$$x = n_1 + \cfrac{1}{n_2 + \cfrac{1}{n_3 + \cfrac{1}{n_4 + \cdots}}} \tag{i}$$

To be more precise, we define $C[n](x) = n + 1/x$ and rewrite equation (i) as

$$x = \lim_{k \to \infty} C[n_1] \circ \cdots \circ C[n_k](\infty).$$

We remark in passing that often one prefers to restrict x to $(0, 1]$, rather than $[1, \infty)$. This can be done by performing the change of coordinates $y = 1/x$. The maps $C[n](x)$, for $n \in \mathbb{N}$, then become the inverse branches of the Gauss map $G(x) = 1/x - [1/x]_*$, where $[z]_*$ denotes the integral part of z.

To use homogeneous coordinates one defines

$$D[n] = \begin{bmatrix} 0 & 1 \\ 1 & n \end{bmatrix}.$$

As

$$D[n] \begin{pmatrix} 1 \\ x \end{pmatrix} \overset{p}{=} \begin{pmatrix} 1 \\ C[n](x) \end{pmatrix},$$

we see that equation (i) is equivalent to

$$\begin{pmatrix} 1 \\ x \end{pmatrix} \overset{p}{=} \lim_{k \to \infty} D[n_1] \cdots D[n_k] \begin{pmatrix} 0 \\ 1 \end{pmatrix}.$$

We now define multidimensional continued fractions in the sense of Jacobi ([**Ja**]) and Perron ([**Pe**]).

We set $\mathbb{N}_0 = \mathbb{N} \cup \{0\}$ and define for each $N \in \mathbb{N}$ and $y = (y_1, \dots, y_N) \in \mathbb{N}_0^N$

$$D[y] = D_N[y] = \begin{pmatrix} 0 & 0 & \cdots & 0 & 1 \\ 1 & 0 & \cdots & 0 & y_1 \\ 0 & 1 & & 0 & y_2 \\ \vdots & & \ddots & & \vdots \\ 0 & 0 & & 1 & y_N \end{pmatrix}.$$

Suppose that for some $N \in \mathbb{N}$ we have a sequence of vectors $y^{(i)} \in \mathbb{N}_0^N$, for $i \in \mathbb{N}$. We then say that a non-negative and non-zero $w \in \mathbb{R}^N$ has *N-ary continued fraction expansion* $[y^{(1)}, \dots, y^{(k)}, \dots]$ if

$$\begin{pmatrix} 1 \\ w \end{pmatrix} \overset{p}{=} \lim_{k\to\infty} D_N[y^{(1)}] \cdots D_N[y^{(k)}] \begin{pmatrix} 0 \\ e_N \end{pmatrix},$$

where $e_N = (0, \dots, 0, 1) \in \mathbb{N}_0^N$.

For $N = 1$ this definition agrees with the one for usual continued fractions as given in the beginning of this section. D_N is a representation of the map

$$C_N[y]\colon \quad \begin{pmatrix} x_1 \\ x_2 \\ \vdots \\ x_N \end{pmatrix} \quad \mapsto \quad \begin{pmatrix} y_1 + \frac{1}{x_N} \\ y_2 + \frac{x_1}{x_N} \\ \vdots \\ y_N + \frac{x_{N-1}}{x_N} \end{pmatrix}$$

in homogeneous coordinates.

Observe that the kth column of a $(N+1) \times (N+1)$ square matrix M equals the $(k-1)$th column of $MD_N[y]$, for $k \in \{2, \dots, N+1\}$. It follows that the (projective) convergence of the last column of $D[y^{(1)}] \cdots D[y^{(k)}]$, as k tends to infinity, implies the convergence of the other columns.

We will need

Proposition 3. *Suppose that M is a $(N+1) \times (N+1)$ matrix. Then there exist vectors $y^{(1)}, \dots, y^{(k)}$ in $\mathbb{N}_0^N$ such that M can be written as a product $M = D_N[y^{(1)}] \cdots D_N[y^{(k)}]$, if and only if M can be reduced to the identity matrix by repeatedly applying the following two operations:*

cyclic permutation of the columns and subtraction of the i^{th} column from the j^{th} column, if $i \neq j$.

The result is constructive and proved in [**Ba1**].

4. A characterization of unique ergodicity

Suppose that $T = T(\pi, \alpha)$ is an i.d.o.c. interval exchange map whose sequence of critical iterates that follows the first critical iterate $n(1)$ for which the corresponding Farey cell is small is $\{n(k)\}_{k=1}^{\infty}$.

Suppose that $S(g, X)$ is the stack that represents T up to the order $n = n(1)$ and that M_k, for $k \in \mathbb{N}$, is the distribution matrix that corresponds to $\mathcal{F}_{n(k)}(T)$. So we have $\alpha = M_1 X$.

We showed that $M_{k+1} = M_1 K_1 K_2 \cdots K_k$, where K_k is a $(2m - 2 \times 2m - 2)$ transition matrix, for $k \in \mathbb{N}$. As each transition matrix differs from the identity matrix only by containing two extra elements 1 off the diagonal in a single row, it follows from proposition 3 that each K_k can be decomposed into a product of matrices $D_N[\cdot]$, with $N = 2m - 3$, as used in the definition of N-ary continued fractions. So there is a sequence $\{y^{(i)}\}_{i=1}^{\infty}$ of vectors in $\mathbb{N}_0^N$ and a subsequence $\{m(k)\}_{k=1}^{\infty}$ of $1, 2, \ldots$, such that

$$M_1 \cdots M_k = D_N[y^{(1)}] \cdots D_N[y^{(m(k))}].$$

We say that $[y^{(1)}, \ldots, y^{(k)}, \ldots]$ is the *N-ary continued fraction associated to $T(\pi, \alpha)$* or $S(g, X)$.

This is formal, as convergence of this continued fraction expansion is not guaranteed in general. Indeed, as we will show next, convergence is equivalent to unique ergodicity of T.

Theorem 4. *Suppose that $T = T(\pi, \alpha)$ is an i.d.o.c. interval exchange map that permutes m intervals. Suppose that n is the first critical iterate for which the corresponding Farey cell $\mathcal{F}_n$ is small and that $S(g, X)$, for some $X \in \mathbb{R}^{2m-2}$, is the stack that represents T up to the order n. Writing $(1, Y) \stackrel{p}{=} X$ we claim:*

T is uniquely ergodic if and only if the $N = 2m - 3$-ary continued fraction expansion associated to T converges to Y.

Proof. We continue to use the notation introduced before the theorem. Suppose first that

$$\lim_{k\to\infty} D_N[y^{(1)}]\cdots D_N[y^{(k)}]e_{N+1} \overset{p}{=} \begin{pmatrix} 1 \\ Y \end{pmatrix},$$

where $e_{N+1} \in \mathbb{R}^{N+1}$ is the $(N+1)$th unit vector. As follows from the last section, we can replace e_{N+1} by any other unit vector $e_i \in \mathbb{R}^{N+1}$. It follows that each column of the product of transition matrices converges projectively to $(1, Y) \overset{p}{=} X$, which, using proposition 2, implies that T is uniquely ergodic.

Conversely, suppose that T is uniquely ergodic. It follows again from proposition 2 that each column of $K_1 \cdots K_k$ converges projectively to X, i.e.

$$\lim_{k\to\infty} D_N[y^{(1)}]\cdots D_N[y^{(m(k))}]e_i \overset{p}{=} X, \tag{ii}$$

for $i \in \{1, \ldots, N+1\}$.

For any $t \in \mathbb{N}$, the map $R_t = D_N[y^{(1)}]\cdots D_N[y^{(t)}]$ maps the positive cone determined by $\{e_1, \ldots, e_{N+1}\}$ to the positive cone C_t determined by $\{R_t e_1, \ldots, R_t e_{N+1}\}$. Equation (ii) implies that for a given $\epsilon > 0$ there is a $k \in \mathbb{N}$, such that some multiple of $R_{n(k)}e_i$ is ϵ-close to X, for every $i \in \{1, \ldots, N+1\}$. This implies that each element of $C_{n(k)}$ has a multiple that is ϵ-close to X. But $R_t(e_{N+1}) \in C_{n(k)}$, for $t \geq n(k)$. This proves that

$$\lim_{t\to\infty} R_t e_N \overset{p}{=} X \overset{p}{=} \begin{pmatrix} 1 \\ Y \end{pmatrix},$$

which is the claimed continued fraction expansion of Y. □

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